conservation equations

conservation of mass:

$$A_t + (UA)_x = 0$$

- A is the cross-sectional area of the tube
- U is the velocity averaged over the cross-section
- x is the distance along the tube
- t is time, we are using subscript notation for partial derivatives

conservation of momentum:

$$U_t + UU_x = -\frac{P_x}{\rho}$$

- P is the hydrostatic pressure averaged over the cross-section
- ρ is the density of blood which is assumed to be constant.

tube law

The conservation equations involve 3 dependent variables, P, U and A. To solve them we need a **tube law:**

$$A(x,t) = A(P(x,t);x)$$

This functional equation just says that the local area is some function of the local pressure which can vary at different locations along the tube x.

$$A_x = A_P P_x + A_x \quad \text{and} \quad A_t = A_P P_t$$

$$\text{where} \quad A_P = \left(\frac{\partial A}{\partial P}\right)_x \quad \text{and} \quad A_x = \left(\frac{\partial A}{\partial x}\right)_P$$

A_P is the local compliance of the artery, *i.e.* the local change in area caused by a change in pressure, which is a measure of the local stiffness of the artery.

canonical equations

Substituting and rearranging,

$$P_t + UP_x + \frac{A}{A_P} U_x = -\frac{UA_x}{A_P}$$

$$U_t + \frac{1}{\rho} P_x + UU_x = 0$$

These equations are written in the canonical form as a system of first order partial differential equations. The coefficients of the x-derivative terms can be written in matrix form

$$\begin{pmatrix} U & \frac{A}{A_P} \\ \frac{1}{\rho} & U \end{pmatrix}$$

which has the eigenvalues

$$\lambda_{\pm} = U \pm \left(\frac{A}{\rho A_P} \right)^{1/2}$$

Wave speed

The square root term in the equation for the eigenvalues has the dimensions of velocity and is, as we will see below, the speed at which changes propagate along the tube; *i.e.* the wave speed. One of the advantages of the method of characteristics is that it gives us an expression for the wave speed in terms of the physical parameters of the problem. Recognising that $D = \frac{A_P}{A}$ is the distensibility of the artery (fractional change in area with a change in pressure), the definition of the wave speed reduces to the expression given by Bramwell and Hill [*Proc. Roy. Soc. London, Series B* (1922) **93**, 298-306]

$$c = \sqrt{\frac{A}{\rho A_P}} = \frac{1}{\sqrt{\rho D}}$$

In general, the wave speed will be a function of both pressure and position in the arteries

$$c = c(P(x,t);x) \qquad \text{since} \qquad A = A(P(x,t);x)$$

This introduces considerable difficulties in wave intensity analysis and so we generally assume that the wave speed at any particular position is a constant, *i.e.* $c = c(x)$. We will make this assumption implicitly in most of the following analysis. It should, however, be kept in mind that it is an approximation to the behaviour of real arteries which generally become stiffer at higher pressures.

Solution by the method of characteristics - 1

Riemann observed that the 'characteristic directions' defined as $\frac{dx}{dt} = \lambda_{\pm} = U \pm c$ play an important role in hyperbolic systems of equations, for which the eigenvalues are real. Along these directions the total derivative with respect to time can be written

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + (U \pm c)\frac{\partial}{\partial x}$$

Substituting into the conservation equations

$$\frac{dP}{dt} - (U \pm c)P_x + UP_x + \rho c^2 U_x = -\frac{UA_x}{A_P}$$

$$\frac{dU}{dt} - (U \pm c)U_x + \frac{1}{\rho}P_x + UU_x = 0$$

Dividing the first equation by $\pm\rho c$ and adding it to the second equation, we obtain the ordinary differential equations

$$\frac{dU}{dt} \pm \frac{1}{\rho c}\frac{dP}{dt} = \mp\frac{UcA_x}{A}$$

where we have used $A_P = \frac{A}{\rho c^2}$. Finally, we can write these equations in terms of the Riemann variables $R_{\pm}$

$$\frac{dR_{\pm}}{dt} = \mp\frac{UcA_x}{A} \qquad \text{where} \qquad R_{\pm} \equiv U \pm \int \frac{dP}{\rho c}$$

This remarkable result says that along the characteristic directions, we can solve for the Riemann variables by solving a simple ordinary differential equation in time.

Solution by the method of characteristics - 2

For the purposes of describing the physical meaning of this rather subtle mathematical result, let us consider the simple case of a uniform vessel. For this case, $A_x = 0$ and so the Riemann variables are constant along the characteristic directions[1].

If there is no velocity in the vessel, then the Riemann variables are constant along lines that propagate upstream and downstream with speed $\pm c$. This justifies our identification of c with the wave speed. If there is a velocity in the vessel, the waves propagate downstream with velocity $U + c$ and upstream with velocity $U - c$. This just says that the waves are convected with the flowing fluid, just as ripples caused by throwing a stone in a river get carried along with the river.

If $U < c$, then one of the waves travels downstream and the other upstream. If $U > c$, then both of the waves propagate downstream and there is no way that changes produced into the vessel at any point can have an effect on the flow upstream. This is what happens in supersonic (or supercritical) flows and explains why subsonic and supersonic flows behave so differently. The convective velocity of blood in the arteries seldom, if ever, exceeds the wave speed and so we will consider only subsonic flows.

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[1] In this case, they are more commonly called Riemann invariants.

Solution by the method of characteristics - 3

If we are interested in what is happening at a particular location x at a particular time t, we simply have to find the waves that intersect at (x, t), determine the value of the Riemann variables $R_\pm$ and then solve for P and U using the above expression for $R_\pm$. Conceptually this is very easy, but in practice it is not so simple. First of all, the path of the wave depends upon the local velocity and the local velocity depends upon the waves arriving there from upstream and downstream. Secondly, the expression for the wave speed depends on the pressure and so we have to solve integral equations to find P and U from the values of $R_\pm$. Making the assumption, discussed above, that c is constant, P and U at (x, t) are simply

$$P = \frac{\rho c}{2}(R_+ - R_-)$$

$$U = \frac{1}{2}(R_+ + R_-)$$

Where $R_\pm$ are the values of the Riemann variables associated with the forward and backward characteristics that intersect at (x, t). Generally, the Riemann variables are given by the boundary conditions that are applied at the inlet and outlet of the vessel. In more complicated circumstances, changes can be imposed upon the vessel, for example, by applying external compression to it at some particular point. In these cases, the Riemann variables are also determined by the conditions imposed everywhere along the vessel, not just at its boundaries.