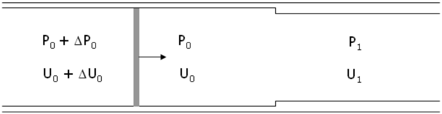
wave reflections - straight vessel - 1

Consider a wavefront propagating forward in a vessel with a discontinuity in its area and/or elastic properties. Assume that the wavefront produces changes in pressure ΔP_0 and velocity ΔU_0.

Before the wave reaches the discontinuity the conditions in the vessel are



Mass and energy conservation at the discontinuity require

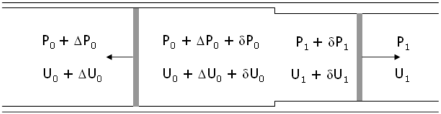
$$U_0 A_0 = U_1 A_1$$

$$P_0 + \tfrac{1}{2}\rho U_0^2 = P_1 + \tfrac{1}{2}\rho U_1^2$$

The second equation follows from the Bernoulli equation and requires that the total pressure be constant across the discontinuity.

wave reflections - straight vessel - 2

After the wavefront encounters the discontinuity, a reflected wave with changes δP_0 and δP_0 will propagate backward in vessel 0 and a transmitted wave with changes δP_1 and δP_1 will propagate forward in vessel 1.



Mass and energy conservation across the discontinuity now require

$$(U_0 + \Delta U_0 + \delta U_0)(A_0 + \Delta A_0 + \delta A_0) = (U_1 + \delta U_1)(A_1 + \delta A_1)$$
$$(P_0 + \Delta P_0 + \delta P_0) + \tfrac{1}{2}\rho(U_0 + \Delta U_0 + \delta U_0)^2 = (P_1 + \delta P_1) + \tfrac{1}{2}\rho(U_1 + \delta U_1)^2$$

We must now solve for the unknown changes $\delta\bullet$ in terms of the known incident changes $\Delta\bullet$.

wave reflections - straight vessel - 3

Expanding the products in the 'after' equation, ignoring terms of second order in the changes and using the 'before' equation to eliminate the $\mathcal{O}(1)$ terms, we are left with two equations of first order in the changes.
Changes in area can be written in terms of changes in pressure using the definition of wave speed $c^2 = \frac{1}{\rho D}$ where $D = \frac{1}{A}\frac{dA}{dP}$

$$dA = \frac{A}{\rho c^2} dP$$

Similarly, the water hammer equations can be used to eliminate changes in velocity in terms of changes in pressure (remembering which waves are forward and which are backward)

$$\Delta P_0 = \rho c_0 \Delta U_0, \qquad \delta P_0 = -\rho c_0 \delta U_0, \qquad \delta P_1 = \rho c_1 \delta U_1$$

The result is two equations for δP_0 and δP_1 in terms of the incident pressure changes ΔP_0

$$\frac{A_0}{c_0}(1 + m_0)\Delta P_0 - \frac{A_0}{c_0}(1 - m_0)\delta P_0 = \frac{A_1}{c_1}(1 + m_1)\delta P_1$$

$$(1 + m_0)\Delta P_0 + (1 - m_0)\delta P_0 = (1 + m_1)\delta P_1$$

where $m_0 = \frac{U_0}{c_0}$ and $m_1 = \frac{U_1}{c_1}$ are the Mach numbers in the two sections of the vessel.

wave reflections - straight vessel - 4

These equations can be solved for the reflection coefficient

$$R = \frac{\delta P_0}{\Delta P_0} = \left(\frac{1+m_0}{1-m_0}\right) \frac{\left(\frac{A_0}{c_0} - \frac{A_1}{c_1}\right)}{\left(\frac{A_0}{c_0} + \frac{A_1}{c_1}\right)}$$

The transmission coefficient follows by substitution into energy equation

$$T = \frac{\delta P_1}{\Delta P_0} = \left(\frac{1+m_0}{1+m_1}\right) + \left(\frac{1-m_0}{1+m_1}\right) R$$

In arterial flows, the blood velocity is normally much smaller than the wave speed, $m << 1$, and so these relationships can be written with excellent accuracy

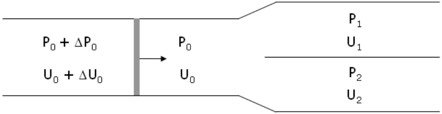
$$R = \frac{\left(\frac{A_0}{c_0} - \frac{A_1}{c_1}\right)}{\left(\frac{A_0}{c_0} + \frac{A_1}{c_1}\right)}$$

$$T = 1 + R$$

These are equivalent to the Kirchoff laws in electrical circuits if we identify $\frac{A}{c}$ as the admittance.

wave reflections - bifurcation - 1

The reflection coefficient for a bifurcation can be obtained following the same steps. Defining the parent vessel as 0 and the daughter vessels as 1 and 2, the conditions before the wavefront arrives at the bifurcation are

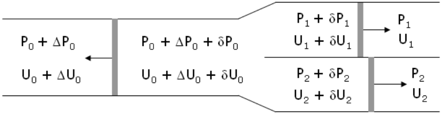


The conservation of mass and energy for the bifurcation is

$$U_0 A_0 = U_1 A_1 + U_2 A_2$$

$$P_0 + \tfrac{1}{2}\rho U_0^2 = P_1 + \tfrac{1}{2}\rho U_1^2 = P_2 + \tfrac{1}{2}\rho U_2^2$$

The first equation says that the net volume flow out of the parent vessel is equal to the net volume flow into the daughter vessels. The energy equation is true along a streamline and so the equation must be true for each daughter vessel individually. After the wavefront has reached the bifurcation, the conditions are



wave reflections - bifurcation - 2

The equivalent conservation equations do not fit on the page, but are straightforward to write. Following the same steps - expanding the nonlinear term, ignoring terms of second order and cancelling terms in the 'after' equation using the 'before' equation - we obtain three first order equations for the changes in the three vessels. Also using the water hammer equation and the definition of the wave speed, we obtain three equations for the three unknowns δP_0, δP_1 and δP_2. These equations have the solution

$$R = \frac{\delta P_0}{\Delta P_0} = \left(\frac{1+m_0}{1-m_0}\right) \frac{\left(\frac{A_0}{c_0} - \frac{A_1}{c_1} - \frac{A_2}{c_2}\right)}{\left(\frac{A_0}{c_0} + \frac{A_1}{c_1} + \frac{A_2}{c_2}\right)}$$

and the transmission coefficients

$$T_n = \frac{\delta P_n}{\Delta P_0} = \left(\frac{1+m_0}{1+m_n}\right) + \left(\frac{1-m_0}{1+m_n}\right) R \qquad n = 1, 2$$

For $m << 1$, these relationships become

$$R = \frac{\left(\frac{A_0}{c_0} - \frac{A_1}{c_1} - \frac{A_2}{c_2}\right)}{\left(\frac{A_0}{c_0} + \frac{A_1}{c_1} + \frac{A_2}{c_2}\right)}$$

$$T_n = 1 + R_n \qquad n = 1, 2$$

wave reflections - bifurcation - 3

These relationships are very important in the arteries. Several things should be noted:

1. When $m << 1$, $+1 \geq R \geq -1$ with $R = 1$ corresponding to the closed end case $A_1 = A_2 = 0$ and $R = -1$ corresponding to the open end case A_1 *or* $A_2 = \infty$.
2. The equations are general and can equally well be used to describe backward wavefronts approaching the bifurcation in either daughter tube. For $m << 1$, the appropriate reflection coefficients can be obtained simply by rotating the indices. For the general case, care must be taken to use the appropriately signed Mach numbers.
3. The special case of $R = 0$ is termed the *well-matched* condition and it means that the wave will pass through the bifurcation without any change. There is some evidence that many arterial bifurcations are well-matched for forward waves.
4. It follows from the form of the equation for R that a bifurcation that is well-matched for waves in one vessel cannot be well-matched for waves approaching in the other vessels. Therefore an arterial bifurcation that is well-matched for forward waves will produce a non-zero reflection coefficient for waves approaching it from either daughter vessel. We will see later that this leads to the phenomenon of *wave trapping* that has important ramifications for arterial haemodynamics.